the burning time t_f does influence the periods of the vibrations, but it does not affect the amplitudes.

As a conclusion the influence of (E'/μ) and $\alpha(\tau)$ on the amplitudes of the dynamic part of the circumferential stress is shown in Fig. 1. Two ablation functions are considered

$$\alpha(\tau) = (1 - \varkappa\tau)^{-1/2} \qquad \text{where} \qquad \varkappa = 1 - (a_0/b)^2 \quad (27)$$

and the linear function

$$\alpha(\tau) = 1 + \lambda\tau \qquad \text{where} \qquad \lambda = (b/a_0) - 1 \quad (28)$$

In both cases (E'/u) has values 10^4, 10^3, and 10^2. The other parameters are chosen as

$$b/a_0 = 3 \qquad h/b = 1/200 \qquad \rho_p/\rho_s = 1/5 \quad (29)$$

and

$$\sigma_i(\tau) = \sigma_0 H(\tau) \quad (30)$$

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A Model for Low-Pressure Extinction of Solid Rocket Motors

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Nomenclature

Rocket-chamber parameters

A_b = propellant burning area
A_t = nozzle-throat area
C_D = nozzle-discharge coefficient
P_c = chamber pressure
P_{cr} = critical chamber pressure for onset of "low-frequency" instability
P_e = extinction chamber pressure
V_c = chamber volume
L^* = V_c/A_t

Propellant parameters

r = propellant burning rate
n = steady-state burning rate pressure exponent, $r = cP^n$
ρ_p = propellant density
P_{DL} = low-pressure stable deflagration limit
M = average molecular weight of gaseous combustion products
T_f = adiabatic flame temperature of propellant
T_g = a temperature parameter that is approximated by the adiabatic flame temperature of the ammonium perchlorate monopropellant flame
k_g = rate constant for the ammonium perchlorate monopropellant flame reactions, $k_g = A_g \exp(-E_g/RT_g)$

Miscellaneous parameters

g = gravitation constant
R = gas constant
t = time
τ = critical time constant

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Introduction

Anderson, Strehlow, and Strand[1] have shown that vacuum firings of solid-propellant rocket motors using regressive burning grains exhibit a low-pressure burning limit that is generally higher than the low-pressure stable deflagration limit of the propellant as determined in a Crawford bomb strand burner. Their studies with ammonium perchlorate composite propellant with varying aluminum content suggest that the extinction pressure (P_e), although independent of burning geometry, is strongly dependent upon Al content and motor L^* (ratio of chamber volume to nozzle-throat area). Their data could be curve-fitted by an expression of the form

$$L^* = A(P_e)^{-\alpha} \quad (1)$$

where the constants A and α varied with Al content.

For nonaluminized propellant, the value of the exponent α was approximately twice the value of the steady-state burning pressure exponent (n) at the pressure P_e. In this case, Eq. (1) becomes almost identical to theoretical expressions derived by Akiba and Tanno,[2] Sehgal and Strand,[3] and Cohen[4] for describing the critical pressure (P_{cr}) for onset of "low-frequency" combustion instability, i.e.,

$$L^* = A'(P_{cr})^{-2n} \quad (2)$$

The theoretical derivations considered the instability as arising from a coupling between the lag of burning-rate response due to the propellant thermal gradient and the lag in exhausting the chamber due to nozzle flow.

The strong resemblance between the theoretical and empirical expressions suggested to these authors[1, 3, 4] that the critical pressure for onset of unstable burning (i.e., P_{cr}) should correspond to the observed extinction pressure (i.e., P_e). However, there are two serious discrepancies that arise from this approach: 1) the value of α for the case of aluminized propellants (8–16%) is much greater than $2n$, and 2) the extinction pressure is zero for an infinite L^*. Since in a Crawford bomb, L^* can be construed as approaching infinity, it might be expected that P_e should have a limiting value approaching the low-pressure stable deflagration limit. Although it has been argued[1, 3, 4] that aluminum or its solid combustion product might in some way affect the propellant thermal response or the nozzle-exhaust time, no adequate quantitative explanation of these discrepancies has been offered.

It is the purpose of this note to offer a possible explanation in terms of a new model for the extinction process. This new approach suggests that Eq. (2) defines criteria for possible temporary extinguishment rather than permanent extinguishment, and, as such, P_{cr} should not be identified with P_e of Eq. (1). In addition, the model enables the derivation of a theoretical P_e vs L^* relationship that differs considerably from the form of Eq. (1).

Model of Rocket Extinction

It is first assumed that solid composite propellants exhibit a low-pressure stable deflagration limit (P_{DL}) that is independent of their extinction under conditions of rocket motor venting. Such phenomena have apparently been noted in strand burners (Crawford bomb) where, below a minimum inert gas bomb pressure, strand burning cannot persist. Low-pressure deflagration limits have been reported as high as 400 psia for some ammonium nitrate propellants[5] and as low as 6 psia for some ammonium perchlorate (AP) propellants. Even ammonium perchlorate monopropellant strands exhibit a P_{DL}.[6]

A detailed discussion of the causes for a P_{DL} in solid composite propellants is outside the scope of the present paper; however it can be noted that Friedman[6] and Nachbar[7] attempted to explain the P_{DL} of AP strands in terms of radiation heat loss from the burning surface. Also, it would appear that aluminum increases the P_{DL} of AP composite propellant.[4]

Second, it is assumed that, during low-pressure burning in a rocket motor, some inherent combustion instability (perhaps related to the low-frequency instability discussed by Sehgal and Strand[3]) causes the instantaneous burning rate r to become zero. The question then arises as to what chamber conditions must be present in order that the instantaneous zero burning rate lead to permanent extinction. On the basis of the first assumption it can be considered that, if the rocket-chamber exhausts to a value of P_{DL} within a critical time period (τ), then the burning rate will remain zero, and the rocket will be extinguished. Now from the rocket chamber mass balance during propellant burning with sonic nozzle exhaust, the time rate-of-change of chamber pressure can be described by [e.g., see Eq. (41) of Ref. 3 where $L^* = V_c/A_t$]

$$\rho_p A_b g r / A_t C_D = P_c + g\ ML^*/C_D\ RT_f(dP_c/dt) \tag{3}$$

Setting $r = 0$, thus gives the rate of chamber exhaust at the instant the burning rate becomes zero. Taking this instant at $t = 0$, the extinction criterion can be stated as $P_c(t = 0) = P_e$ when, during the time interval $0 < t \leq \tau$, the chamber pressure is reduced to the value P_{DL}. Utilizing the preceding expression with $r = 0$ for $P_c(t)$ over the appropriate time interval, the minimum extinction requirement becomes

$$\int_{Pe}^{P_{DL}} \frac{d\,P_c}{P_c} = -\frac{C_D RT_f}{g\ ML^*}\int_0^{\tau} dt \tag{4}$$

or

$$P_e = P_{DL}\exp\left[\frac{C_D RT_f \tau}{g\ ML^*}\right] \tag{5}$$

It is interesting to note that Eq. (5) is considerably different from the empirical form used by Anderson et al. [Eq. (1)], and that, if τ is finite and independent of L^*, P_e approaches P_{DL} as L^* approaches infinity. Thus, the new model of extinction would appear to resolve one of the discrepancies as noted in the introduction.

It now remains to describe the physical nature of the critical time constant τ. One possibility that arises is that τ relates to the kinetics of the propellant flame reactions. This might be reasonable from the view point that, at the instant ($t = 0$) when r becomes zero, there is no mass efflux from the surface, whereas the propellant surface temperature is relatively unaltered from its value when $r \neq 0$. At $t > 0$, there is a positive mass efflux with the outflowing gases requiring τ sec to react and to supply heat back to the propellant surface. However, if during these τ sec the chamber pressure drops to P_{DL}, then the gas-phase reactions become quenched, and hence the extinction becomes permanent. In a sense, this extinction process can be likened to the process by which rarefaction waves quench detonation reactions in explosives, giving rise to critical-diameter phenomena.[8]

For second-order flame reactions, the half-life of the reactions is directly proportional to the reciprocal of the first power of the chamber pressure[9]; hence, identifying τ with the half-life of the flame reactions yields

$$\tau = \beta/P_c \tag{6}$$

For the case where the ammonium perchlorate gas-phase redox reaction controls the flame kinetics ("thermal layer" theory of combustion[10]), the constant β can be equated to $2\ RT_g/k_g$, where k_g refers to the rate constant of the $NH_3 + HClO_4$ flame reaction at the temperature T_g, approximately the adiabatic flame temperature for this reaction.

Using the previous expression for τ, Eq. (4) becomes

$$P_e = P_{DL}\exp[E/P_e L^*] \tag{7}$$

where $E = C_D RT_f \beta/gM$ is a constant for any given propellant composition.

To test the possible validity of Eq. (7), the data of Anderson et al.[1] have been plotted as $\log P_e$ vs $(L^*P_e)^{-1}$ in Fig. 1. It is readily seen that the theoretical expression offers a reasonably good fit to the data. Also, it is noted that for JPL 534 and 540 (~0% Al), extrapolation to infinite L^* gives a value of $P_{DL} = 8$ psia in good agreement with the reported[1] value of $P_{DL} = 6$ psia.

Similar extrapolations for the propellants containing 8% and 16% Al yield higher values of P_{DL} (29 psia and 42 psia, respectively) as might be expected. Unfortunately, Anderson et al. did not report P_{DL} values for these propellants, so that a more precise comparison of results can not be made at this time.

It is interesting to calculate the expected order of magnitude of E when $\beta = 2\ RT_g/k_g$. Although the value of k_g to be used in this expression is quite uncertain, it can be estimated from previous work[10] that a reasonable order of magnitude is $k_g \simeq 10^9$ cm³/mole-sec. Using this value and reasonable estimates of the other parameters,† E is approximately 10^2 in.-psia. The slopes of the curves of Fig. 1 (i.e., experimental E) lie in the range of 1.6×10^3 to 4.0×10^3 in.-psia (increasing with decreasing Al content), which differ from the theoretical calculated E by less than a factor of 100. In view of the gross uncertainty in k_g, this comparison of theoretical and experimental E can be considered as favorable but not necessarily significant. However, assuming the calculations to be reasonably valid, it would suggest that increasing Al content may cause an increase in the effective temperature at which the bulk of the AP redox flame reactions occur (i.e., T_g).‡ Alternatively, it may be speculated, that the nozzle discharge coefficient (C_D) decreases as the amount of solids in the exhaust increases. In either case, there is insufficient data at this time to allow for a more precise account of the effect of aluminum on the L^* vs P_e relationship.

Conclusions

The model of solid rocket extinction presented here would appear to resolve some of the difficulties that are apparent when attempting to identify criteria for "low-frequency" combustion instability with low-pressure extinction criteria.

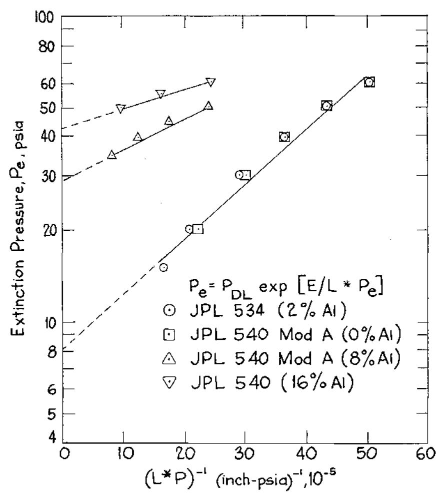

Fig. 1 Variation of extinction pressure with L^* (data points determined from smoothed curve of form $\log L^* = \log A - \alpha \log P_e$, which was fit to experimental data by Anderson, Strehlow and Strand[1]).

† $C_D = 7 \times 10^{-3}$ (sec)$^{-1}$, $R = 8.31 \times 10^7$ erg/deg-mole $=$ 82 cm³-atm/deg-mole, $M = 30$ g/mole, $g = 980$ cm/sec², $T_f = 3000$°K, and $T_g = 1400$°K.

‡ In the case being considered, $E = 2C_D R^2 T_f T_g/gMk_g$ with $k_g = A_g \exp(-E_g/RT_g)$. The exponential dependence of k_g on temperature in the denominator will outweigh the linear dependence in the numerator.

The model also enables a treatment of extinction in terms of the kinetics of the propellant flame reactions, yielding a new form of equation for relating motor L^* and extinction pressure which is in satisfactory agreement with the available data. However, it is clear that the existing data encompasses only a narrow part of the spectrum of experimental results required for an adequate test of this or any other theoretical treatment.

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Circular Sandwich Plate under Radial Compression and Thermal Gradient

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SINCE the early work of Reissner[1] on sandwich construction, for which the effect of transverse shear deformation is known to be appreciable, the bending of circular sandwich plates has become the subject of numerous investigations. Ericksen[2] developed formulas for the deflection and stresses in a circular sandwich plate under normal load. A general treatment of the axisymmetric problem of bimetallic circular plates including thermal effects has been given by Grigoliuk.[3] Zaid[4] extended Reissner's work to include the bending rigidities of the facings. Thermal stresses in laminated circular plates have been investigated by Vinson.[5] More recently Bruun[6] has analyzed the thermal deflection problem of a circular sandwich plate. None of these works, however, has considered the combined effect of radial compression and a thermal gradient. Accordingly, the basic displacement equations in polar coordinates with axial symmetry derived elsewhere by Huang[7] for a circular sandwich plate (having a soft isotropic core and isotropic facings of different thicknesses and materials) under radial compression and thermal gradient and including the bending rigidity of the facings are here solved for certain special cases. These are

$$\left.\begin{aligned}
&\frac{E't'}{1-\mu'^2}\left(r\frac{d^2u'}{dr^2}+\frac{du'}{dr}-\frac{u}{r}\right)- \\
&\qquad r\bar{G}\left(\frac{u'-u''}{\bar{t}}+\hat{t}\frac{dW}{dr}\right)=r\frac{dN'}{dr} \\
&\frac{E''t''}{1-\mu''^2}\left(r\frac{d^2u''}{dr^2}+\frac{du''}{dr}-\frac{u''}{r}\right)+ \\
&\qquad r\bar{G}\left(\frac{u'-u''}{\bar{t}}+\hat{t}\frac{dw}{dr}\right)=r\frac{dN''}{dr}
\end{aligned}\right\} \tag{1}$$

$$\begin{aligned}
&D\nabla^2\nabla^2 w-\bar{t}\hat{t}\bar{G}\frac{1}{r}\frac{d}{dr}\left[r\left(\frac{u'-u''}{\bar{t}}+\hat{t}\frac{dw}{dr}\right)\right]-q- \\
&\qquad N_r\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)=-\nabla^2 M
\end{aligned} \tag{2}$$

where, in addition to the quantities shown in Fig. 1,

$$\nabla^2\equiv\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\qquad N_r=N_r'+N_r''$$

$$M=M'+M''\qquad q=q'+q''$$

$$\begin{Bmatrix}N'\\M'\end{Bmatrix}=\frac{E'\alpha'}{1-\mu'}\int_{-t'/2}^{t'/2}T'\begin{Bmatrix}1\\z'\end{Bmatrix}dz'$$

$$\begin{Bmatrix}N''\\M''\end{Bmatrix}=\frac{E''\alpha''}{1-\mu''}\int_{-t''/2}^{t''/2}T''\begin{Bmatrix}1\\z''\end{Bmatrix}dz''$$

$$D=\frac{E't'^3}{12(1-\mu'^2)}+\frac{E''t''^3}{12(1-\mu''^2)}\qquad \hat{t}=\frac{\bar{t}+(t'+t'')/2}{\bar{t}}$$

and u', u'', N_r', N_r'', q', q'', E', E'', μ', μ'', α', α'', T', and T'' denote, respectively, the radial displacements, radial membrane forces, external transverse normal loads, Young's moduli, Poisson's ratios, coefficients of thermal expansion, and temperature in the facings; $\bar{G}$ is the shear modulus of the core. The solution of Eq. (1) is

$$\left.\begin{aligned}
u'&=\frac{1-\mu'^2}{E't'\hat{t}\bar{t}}\left[D\frac{dw}{dr}+\tilde{N}_r\frac{1}{r}\int rwdr+\right. \\
&\qquad\frac{1}{r}\int(M+\hat{t}\bar{t}N')rdr- \\
&\qquad\qquad\left.\frac{1}{r}\int rdr\int\frac{dr}{r}\int qrdr+\frac{c_2'r}{2}\right] \\
-u''&=\frac{1-\mu''^2}{E''t''\hat{t}\bar{t}}\left[D\frac{dw}{dr}+\tilde{N}_r\frac{1}{r}\int rwdr+\right. \\
&\qquad\frac{1}{r}\int(M-\hat{t}\bar{t}N'')rdr- \\
&\qquad\qquad\left.\frac{1}{r}\int rdr\int\frac{dr}{r}\int qrdr+\frac{c_2''r}{2}\right]
\end{aligned}\right\} \tag{3}$$

where $\tilde{N}_r=-N_r$ and C_2', C_2'' are integration constants the rest of which has been shown to vanish by Zaid.[4] Because of Eq. (3), but with $q=0$ and constant N', N'', and M, Eq. (2) now becomes

$$(\nabla^2-m_1^2)(\nabla^2-m_2^2)w=N_1C_2/D \tag{4}$$

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